GOVERNMENT OF ANDHRA PRADESH

COMMISSIONERATE OF COLLEGIATE EDUCATION

LAPLACE TRANSFORMS - III

DIVISION BY ‘t’

MATHEMATICS

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LAPLACE TRANSFORMS – DIVISION BY ‘t’

Learning Objectives

Students will be able to recognize the following properties of the Plane:

- Understand the definition and properties of Laplace transformations.
- Get an idea about division by t.
- Understand Laplace transforms of standard functions.
- Get the knowledge of application of Laplace transformations.

Laplace Transform :

If the kernel K(p,t) is defined as K (p, t) = $\begin{cases} 0 \; for \; t < 0 \\ e^{-pt} \; for \; t \geq 0 \end{cases}$ then f(p)=$\int_0^\infty e^{-pt} F(t)dt$ is called the Laplace Transform of the function F(t) and is also denoted by L{F(t)} or $\bar{F}$(p) .

Therefore L{F(t)} = f(p) = $\int_0^\infty e^{-pt} F(t)dt$

Laplace Transform of some elementary functions

1. $L\{k\} = \frac{k}{p} \quad (p > 0)$
2. $L\{t^n\} = \frac{\gamma(n+1)}{p^{n+1}}$, $p > 0$ and n is any real number .
3. $L\{t^n\} = \frac{n!}{p^{n+1}}$, $p > 0$ and n is a positive integer .
4. $L\{e^{at}\} = \frac{1}{p-a}$, if $p > a$.
5. $L\{e^{-at}\} = \frac{1}{p+a}$, if $p > -a$.
6. $L\{\sin at\} = \frac{a}{p^2+a^2}$, $p > 0$.
7. $L\{\cos at\} = \frac{p}{p^2+a^2}$, $p > 0$.
8. $L\{\sinh at\} = \frac{a}{p^2-a^2}$, $p > |a|$.
9. $L\{\cosh at\} = \frac{p}{p^2-a^2}$, $p > |a|$.

Laplace Transformation – Division by ‘ t ‘ :

If $L[\,F(t)] = f(p)$ then $L\left[\frac{F(t)}{t}\right] = \int_p^{\infty} f(p)\,dp$, provided the integral exists.

Proof: Given $f(p) = L[\,F(t)] = \int_0^{\infty} e^{-pt}\,F(t)dt$

Integrating both sides w.r.t.’p’ from p to ∞, we get

$$\int_p^{\infty} f(p)dp = \int_p^{\infty}[\int_0^{\infty} e^{-pt}\,F(t)dt]\,dp$$

The order of integration in the double integral can be interchanged since ‘p’ and ‘t’ are independent variables.

$$\therefore \int_p^{\infty} f(p)dp = \int_0^{\infty} dt \int_p^{\infty} e^{-pt}\,F(t)dp$$

$$= \int_0^{\infty} F(T)dt \int_p^{\infty} e^{-pt}\,dp$$

$$= \int_0^\infty F(T)dt \left[\frac{e^{-pt}}{-t}\right]_p^\infty$$

$$= \int_0^\infty \frac{F(t)}{t} e^{-pt} dt$$

$$= L\left[\frac{F(t)}{t}\right]$$

Hence $L\left[\frac{F(t)}{t}\right] = \int_p^\infty f(p)\, dp.$

Ex 1: Find $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Sol: Let $F(t) = e^{-at} - e^{-bt}$.

$$Then\ L\{e^{-at} - e^{-bt}\} = \frac{1}{p+a} - \frac{1}{p+b} = f(p)$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \left[\frac{F(t)}{t}\right] = \int_p^\infty f(p)\,dp$$

$$= \int_p^\infty \left(\frac{1}{p+a} - \frac{1}{p+b}\right) dp$$

$$= (\log(p+a) - \log(p+b))_p^\infty$$

$$= \log\left(\frac{p+a}{p+b}\right)_p^\infty$$

$$\therefore L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \log\left(\frac{p+b}{p+a}\right)$$

Ex 2: Evaluate $L\left(\frac{\sin at}{t}\right)$

Sol: We know that L{ f(t) } = L { sin at } $= \frac{\mathbf{a}}{\mathbf{p^2+a^2}}$, p > 0 = f(P)

and $L\left[\frac{F(t)}{t}\right] = \int_p^{\infty} f(p)\, dp$

Therefore $L\left(\frac{\sin at}{t}\right) = \int_p^{\infty} \frac{\mathbf{a}}{\mathbf{p^2+a^2}}\, dp$

$$= \left(tan^{-1}\frac{p}{a}\right)_p^{\infty}$$

$$= \frac{\pi}{2} - tan^{-1}\frac{p}{a}$$

$$\therefore L\left(\frac{\sin at}{t}\right) = cot^{-1}\frac{p}{a} \ .$$

Ex 3: Show that $L\left(\frac{\cos at}{t}\right)$ does not exist.

Sol: We know that L{ F(t) } = L { cos at } $= \frac{\mathbf{p}}{\mathbf{p^2+a^2}}$, p > 0 = f(P)

and $L\left[\frac{F(t)}{t}\right] = \int_p^\infty f(p)\, dp$

Therefore $L\left(\frac{\cos at}{t}\right) = \int_p^\infty \frac{\mathbf{p}}{\mathbf{p^2+a^2}}\, dp$

$$= \frac{1}{2}\log(p^2 + a^2)_p^\infty$$

$$= \frac{1}{2}(\log \infty - \log(p^2 + a^2))$$

$$\therefore L\left(\frac{\cos at}{t}\right) = does\ not\ exists.$$

Ex 4: Evaluate $L\left(\frac{1-\cos at}{t}\right)$.

Sol: We know that $L\{F(t)\} = L\{1-\cos at\}$

$$= L\{1\} - L\{\cos at\}$$

$$= \frac{1}{p} - \frac{p}{p^2+a^2} = f(p)$$

and $L\left[\frac{F(t)}{t}\right] = \int_p^\infty f(p)\, dp.$

So $L\left\{\frac{1-\cos at}{t}\right\} = \int_p^\infty \left(\frac{1}{p} - \frac{p}{p^2+a^2}\right) dp$

$$= \{\log p - \frac{1}{2}(\log(p^2+a^2)\}_p^\infty$$

$$= \frac{1}{2}\{2\log p - (\log(p^2+a^2)\}_p^\infty$$

$$= \frac{1}{2}\left\{\log\left(\frac{p^2}{p^2+a^2}\right)\right\}_p^\infty$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{p^2}{p^2+a^2}\right)\right\}$$

$$= -\frac{1}{2}\log\left(\frac{p^2}{p^2+a^2}\right)$$

$$= log\left(\frac{p^2}{p^2+a^2}\right)^{\frac{-1}{2}}$$

$$\therefore L\left\{\frac{1-\cos at}{t}\right\} = log\sqrt{\frac{p^2+a^2}{p^2}}$$

Ex 5: Find the Laplace transform of $\left\{\frac{1-\cos t}{t}\right\}$

Ans: Put a = 1 in the above problem we get $L\left\{\frac{1-\cos t}{t}\right\} = log\sqrt{\frac{p^2+1}{p^2}}$

Ex 6: Find $L\left\{\frac{\cos at-\cos bt}{t}\right\}$,

Sol: We know that $L\{F(t)\} = L\{\cos at - \cos bt\}$

$$= \mathrm{L}\{\cos at\} - \mathrm{L}\{\cos bt\}$$

$$= \frac{p}{p^2+a^2} - \frac{p}{p^2+b^2} = f(p)$$

and $L\left[\frac{F(t)}{t}\right] = \int_p^\infty f(p)\, dp.$

So $L\left\{\frac{\cos at-\cos bt}{t}\right\} = \int_p^\infty \left(\frac{p}{p^2+a^2} - \frac{p}{p^2+b^2}\right) dp$

$$= \left\{\frac{1}{2}\log(p^2 + a^2) - \frac{1}{2}(\log(p^2 + b^2)\right\}_p^\infty$$

$$= \frac{1}{2}\log\left(\frac{1+\frac{a^2}{p^2}}{1+\frac{b^2}{p^2}}\right)_p^\infty$$

$$= \frac{1}{2}\left(log1 - \log\left(\frac{p^2+a^2}{p^2+b^2}\right)\right)$$

$$\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\left(\frac{p^2+b^2}{p^2+a^2}\right)$$

Ex 6: Find the Laplace transform of $\left\{\frac{\cos t - \cos 2t}{t}\right\}$

Sol: Put a = 1 and b = 2 in the above problem we get

$$L\left\{\frac{\cos t - \cos 2t}{t}\right\} = \frac{1}{2}\log\left(\frac{p^2+4}{p^2+1}\right)$$

Ex 7: Find $L\left(\frac{1-\cos t}{t^2}\right)$

Sol: We know that $L\{F(t)\} = L\{1-cos\ t\} = L(1) - L(cos\ t) = \frac{1}{p} - \frac{p}{p^2+1} = f(p)$

and $L\left[\frac{F(t)}{t}\right] = \int_p^\infty f(p)\,dp.$

So $L\left\{\frac{1-\cos t}{t}\right\} = \int_p^\infty (\frac{1}{p} - \frac{p}{p^2+1})\,dp$

$$= \ \{\log p - \frac{1}{2}(\log(p^2+1)\}_p^\infty$$

$$= \frac{1}{2}\ \{2\log p - (\log(p^2+1)\}_p^\infty$$

$$= \frac{1}{2}\left\{\log\left(\frac{p^2}{p^2+1}\right)\right\}_p^\infty$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{p^2}{p^2+1}\right)\right\}$$

$$= -\frac{1}{2}\log\left(\frac{p^2}{p^2+1}\right)$$

$$= log\left(\frac{p^2}{p^2+1}\right)^{\frac{-1}{2}}$$

$$\therefore L\left\{\frac{1-\cos t}{t}\right\} = log\sqrt{\frac{p^2+1}{p^2}}$$

$$\therefore L\left\{\frac{1-\cos t}{t^2}\right\} = \int_p^\infty \frac{1}{2}\log\left(\frac{P^2+1}{p^2}\right)dp$$

$$= \frac{1}{2}\int_p^\infty \{\log(p^2+1) - logp^2\}dp$$

$$= \frac{1}{2}\int_p^\infty \{\log(p^2+1) - 2logp\}\,1dp$$

$$= \frac{1}{2}(\log(p^2 + 1) - 2logp)\,{}_p^{\infty} - \frac{1}{2}\int_p^{\infty}\left(\frac{2p}{p^2+1} - \frac{2}{p}\right)p\,dp$$

$$= \left(\frac{p}{2}\log\left(\frac{P^2+1}{p^2}\right)\right)_p^{\infty} + \int_p^{\infty}\frac{1}{p^2+1}\,dp$$

$$= \left(\frac{p}{2}\log\left(1 + \frac{1}{p^2}\right)\right)_p^{\infty} + (tan^{-1}p)_p^{\infty}$$

$$= -\frac{p}{2}\log\left(1 + \frac{1}{p^2}\right) + \left(\frac{\pi}{2} - tan^{-1}p\right)$$

$$= cot^{-1}p - \frac{p}{2}\log\left(1 + \frac{1}{p^2}\right)$$

$$\therefore L\left\{\frac{1-\cos t}{t^2}\right\} = cot^{-1}p - \frac{p}{2}\log\left(1 + \frac{1}{p^2}\right)$$

Reference Books:

- Introduction to Applied Mathematics by Gilbert Strang, Cambridge Press
- Laplace and Fouries transforms by Dr.J.K. Goyal and K.P. Guptha, PragathiPrakashan, Meerut.
- The Laplace Transform: Theory and Applications by Joel L.Schiff
- The Laplace Transform by David Vernon Widder
- Student’s Guide to Laplace Transforms (Student's Guides) by Daniel Fleisch.

Thank you!